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**Generalized Higher Gauge Theory and M5-brane dynamics.
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We give a review of truncated L_∞ algebras, as used in the study of higher gauge theory. These structures are believed to hold the correct properties to adequately describe gauge theory of extended objects. We discuss how to construct topological higher-gauge-invariant theories and how their solutions relate to multisymplectic geometries. We also show how Courant algebroids fit into this formalism, so as to be able to study higher gauge theory on generalized geometric bundles, *i.e.* on $T\Sigma \oplus T^*\Sigma$, for some space-time Σ . We will see that via this formalism we can match and explain a recently proposed M5-brane model, arrived at in a more heuristic way, whose field content seemed difficult to interpret but finds a natural motivation in this framework.

Keywords: Strong homotopy algebras, Lie n -algebras, higher gauge theory models, generalized geometry, Courant algebroids, M5-brane effective dynamics

1. Introduction

In the spirit of “Higher structures in String and M-theory”, we will attempt here a relatively self-contained review of some techniques and an intriguing explicit example of how higher gauge theory connects with other novel mathematics introduced for the study of extended objects.

We know that if we assume the fundamental entities in physics not to be point-like anymore, we will have to deal with constructing gauge theories over higher dimensional world-volumes. It has been known for a long time, however, that it is impossible to make an action (say a Yang-Mills type model) reparametrization invariant, if we allow for p -form gauge connections, for $p > 1$, that take values in a non-abelian symmetry algebra (for an explicit, physical calculation, see for instance¹). As was explained

in more detail in the lectures during this workshop, the issue boils down to our inability to uniquely define “higher holonomy” for $(d > 1)$ -dimensional volumes, as it is impossible to uniquely assign something like a surface- (or volume) ordering over which to integrate higher p -form connections. There are obviously many ways to, say, move bits of paths along a surface, before gluing them back together, that might not necessarily lead to the same result when integrated over: one needs to impose that all possibilities, that cover the same world-volume, lead to the same holonomy assignment. As it turns out, requiring this sort of equivalence, corresponds precisely to the set of defining rules of higher categories. This may not be surprising, as we are really just restating, in different languages, our need for a certain amount of associativity between different “products” in a very general set-up.

The power of category theory comes in at the next step: since we can use functors F to map between categories $\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2$, and these have to preserve the properties of all the maps of both \mathcal{C}_1 and \mathcal{C}_2 , the various components of F will in turn have to satisfy a very specific set of rules. For our intents and purposes, the categories to map between are usually a space-time manifold on the one side, and an interior symmetry group-like structure on the other. The functor F between them will contain the necessary information about the gauge connections’ properties, according to how high the two categories are. In the familiar example of ordinary gauge theory, we would be mapping between the *path 1-groupoid* associated to a manifold (where the objects are points and morphisms are paths, or world-lines, between them, roughly speaking) and a gauge group G , which in turn can be seen as a *1-groupoid*, with just the one object G and the group elements as morphisms. One then has a functor that associates a group element to each closed path γ on the manifold, via the familiar holonomy map:

$$F: \gamma \longrightarrow \mathcal{P} \left(\exp \int_{\gamma} A \right) \in G ,$$

where \mathcal{P} stands for the path-ordering and A is a 1-form valued in the Lie algebra of G . Requiring that F be a functor (that is, preserves the associative morphisms of both underlying categories) in fact imposes on A all the properties that define a connection (see² for a pedagogical introduction).

Moving up one step, world-surfaces would be described, roughly speaking, by the *2-category* consisting of points, paths and surfaces between them, while its gauge symmetry structure would be expected to be a *Lie 2-group*. These higher categories are endowed with *2-morphisms*, as well as morphisms, whose various mixed composition rules also have to be pre-

served by what will here be a *2-functor*. A consistent analogue of the holonomy map now has to be constructed out of a pair of gauge fields $(A_\mu, B_{\mu\nu})$, a 1- and a 2-form respectively, each valued in one of the two components of the *Lie 2-algebra* corresponding to the Lie 2-group in question. These structures, in turn, have to satisfy their own specific set of rules, that we will elucidate in the first section below. Furthermore, A and B can be shown to have to satisfy the so-called *vanishing fake curvature condition*, *i.e.* that $F - t(B) = 0$, where t is a map between the two components of the 2-algebra. For details on this see C. Sämann's lectures, part of these same Proceedings, or^{2,3}.

In what follows we will not delve into the details of this motivation. We will however present the definitions and salient properties of the mathematical tools that are used for higher gauge theory. We will explain how one can describe higher Lie algebras, as well as some examples of them that have already been encountered in physics. We will also show how one can construct general topological models based on higher symmetry structures and how these relate to a higher analogue of the Poisson algebra on symplectic manifolds, *i.e.* to the Lie n -algebras on n -plectic spaces.

In section 4 we will show explicitly how Courant algebroids, as they appear in generalized complex geometry and in double field theory, come equipped with their own Lie 2-algebras. In fact, we will see that they contain the 2-algebra structure of a 2-plectic manifold. We then consider the space-time side of our connection functor to be already generalized, to $T\Sigma \oplus T^*\Sigma$, while allowing for a general 2-algebra on the internal symmetry side. Finally, in section 5.2 we will show how the flatness conditions on the higher connections in this setup precisely reproduce the equations of motion proposed recently, in⁴, for the effective dynamics of M5-branes. There, a different generalization of Lie algebras is used, but we will again see that it is just another example of a special, *strict*, 2-algebra.

2. Mathematical tools

Let us first introduce the arsenal of mathematical tools that will be needed in the following sections. Here we will give two equivalent definitions of truncated strong homotopy algebras (denoted L_∞ -algebras), or Lie n -algebras, each of which we will see to be useful in different contexts.

2.1. Lie n -algebras

Definition 2.1. An L_∞ -algebra or *strong homotopy Lie algebra* is a graded vector space $L = \oplus_i L_i$ endowed with n -ary multilinear totally antisymmetric products μ_n , $n \in \mathbb{N}^*$, of degree $(2 - n)$, that satisfy homotopy Jacobi identities, cf. ⁵⁻⁷. These identities read as

$$\sum_{i+j=n} \sum_{\sigma} \chi(\sigma; l) (-1)^{i \cdot j} \mu_{j+1}(\mu_i(l_{\sigma(1)}, \dots, l_{\sigma(i)}), l_{\sigma(i+1)}, \dots, l_{\sigma(i+j)}) = 0$$

for all $n \in \mathbb{N}^*$, where the sum over σ is taken over all (i, j) unshuffles. A permutation σ of $i + j$ elements is called an (i, j) -unshuffle, if the first i and the last j images of σ are ordered: $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(i+j)$. Moreover, the *graded Koszul sign* $\chi(\sigma; l)$, for $l = (l_1, \dots, l_n)$ and $l_i \in L$ is defined via the equation

$$l_1 \wedge \dots \wedge l_n = \chi(\sigma; l) l_{\sigma(1)} \wedge \dots \wedge l_{\sigma(n)}$$

in the free graded algebra $\wedge(l_1, \dots, l_n)$, where \wedge is considered graded anti-symmetric.

Truncated strong homotopy Lie algebras are concentrated in degrees $(-n + 1), \dots, 0$, so that $L_i = *$ for $i \notin [-n + 1, 0]$. Consequently, because of their grading, the μ_k products will vanish for $k > (n + 1)$. These truncated L_∞ algebras are believed to be categorically equivalent to semi-strict Lie n -algebras^a, and are therefore expected to be the correct infinitesimal symmetry structure for gauge theories of extended objects.

Specifically, we will be interested in the case of semi-strict Lie 2-algebras, which will be given by the 2-term real vector-space complex

$$L : V \xrightarrow{\mu_1} W \xrightarrow{\mu_1} 0,$$

where here $L_{-1} \equiv V$ and $L_0 \equiv W$.

The $n = 2$ example is of interest when studying, for instance, gauge theory over the world-surface of a 1-dimensional object, such as a string. In particular, when searching for an effective gauge theory as seen by an M2-brane intersecting a stack of extended objects, for which we expect a non-abelian internal symmetry group⁸.

^aThis has as yet only been proven for $n = 2$, nonetheless we will continue to use both terms interchangeably for the remainder of this paper.

The homotopy product μ_1 has degree 1 and squares to zero, while the grading also imposes that

$$\begin{aligned}\mu_1(w) &= 0, \quad \mu_2(v_1, v_2) = 0, \\ \mu_3(v_1, v_2, v_3) &= \mu_3(v_1, v_2, w) = \mu_3(v_1, w_1, w_2) = 0.\end{aligned}$$

The 2-algebra's non-vanishing higher products satisfy the following higher Jacobi identities:

$$\begin{aligned}\mu_1(\mu_2(w, v)) &= \mu_2(w, \mu_1(v)), \quad \mu_2(\mu_1(v_1), v_2) = \mu_2(v_1, \mu_1(v_2)), \\ \mu_1(\mu_3(w_1, w_2, w_3)) &= -\mu_2(\mu_2(w_1, w_2), w_3) - \text{cyclic}(w_1, w_2, w_3), \\ \mu_3(\mu_1(v), w_1, w_2) &= -\mu_2(\mu_2(w_1, w_2), v) - \text{cyclic}(w_1, w_2, v),\end{aligned}$$

where $v_i \in V$ and $w_i \in W$ have degrees -1 and 0 respectively.

The equalities above show how the elements in V and W mix in a non-trivial way: indeed, only when $\mu_3 = 0$, we are just describing a *differential crossed module* of actual Lie algebras, given by (W, V, μ_1, α) , where the action α of W on V is given by the product $\mu_2(w, v)$. It is clear from the last two lines that, for non-vanishing μ_3 , the Jacobi identity of traditional Lie algebras is violated in a controlled way, by a μ_1 -exact term.

We have one further identity coming from definition (2.1):

$$\begin{aligned}\mu_2(\mu_3(w_1, w_2, w_3), w_4) &- \mu_2(\mu_3(w_4, w_1, w_2), w_3) + \mu_2(\mu_3(w_3, w_4, w_1), w_2) \\ &- \mu_2(\mu_3(w_2, w_3, w_4), w_1) = \\ \mu_3(\mu_2(w_1, w_2), w_3, w_4) &- \mu_3(\mu_2(w_2, w_3), w_4, w_1) + \mu_3(\mu_2(w_3, w_4), w_1, w_2) \\ &- \mu_3(\mu_2(w_4, w_1), w_2, w_3) - \mu_3(\mu_2(w_1, w_3), w_2, w_4) - \mu_3(\mu_2(w_2, w_4), w_1, w_3),\end{aligned}$$

specifying how the ternary product μ_3 mixes with μ_2 .

Just like with Lie algebras and the Chevalley-Eilenberg complex, here too we have an equivalent dual description of the structure, via NQ -manifolds (as introduced in⁹):

Definition 2.2. An NQ -manifold is a \mathbb{N} -graded manifold

$$\mathcal{M} = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots,$$

endowed with a degree 1, nilpotent, differential operator Q :

$$Q = \sum_i \frac{1}{i!} m_{C_1 \dots C_i}^B Z^{C_1} \dots Z^{C_i} \frac{\partial}{\partial Z^B}, \quad (1)$$

where $\{Z^{C_k}\}$ are coordinates of degree $[Z^{C_k}]$ parametrizing \mathcal{M} , and $\sum_{l=1}^i [Z^{C_l}] = [Z^B] + 1$.

Note that in the above definition the components M_i have positive degree, while in the previous L_∞ -algebra definition we started with a complex of negatively graded vector spaces. This is a matter of convention: one could easily redefine the L_i components to have positive grading, and μ_1 to map “downward”, but we prefer to stick to these choices so as to stay in line with what is most commonly used in the literature.

Requiring the nilpotency of the general operator Q in (1) yields a set of conditions on the “structure coefficients” $m_{C_1 \dots C_i}^B$, which are just the dual equivalent of the higher Jacobi identities we obtain from eqn. (2.1). Indeed, if we take a manifold \mathcal{M} which has no degree zero component, $M_0 = *$, and make the identification

$$\mu_k(\tau_{C_1}, \dots, \tau_{C_k}) = m_{C_1 \dots C_k}^B \tau_B ,$$

where τ_A is a basis for \mathcal{M} , and we assign degrees $[\tau_A] = 1 - [Z^A]$ to adjust for the inversion of the grading between the L_∞ complex and the NQ -manifold definitions mentioned above. Requiring $Q^2 = 0$ will translate to the correct higher homotopy structure for the μ_k . When $M_0 \neq *$, the construction of the homotopy products is more subtle, requiring the use of derived coalgebra techniques. In this case one is describing *Lie n -algebroids*, which are just the categorification of traditional Lie algebroids.

Let us consider again the 2-algebra example: take $\mathcal{M} = W[1] \oplus V[2]$, with coordinates $\{w^a, v^i\}$ of degrees $(1, 2)$. A general degree-1 differential operator Q is given by

$$Q = \left(-\frac{1}{2} f_{bc}^a w^b w^c - t_i^a v^i \right) \frac{\partial}{\partial w^a} + \left(\frac{1}{6} h_{abc}^i w^a w^b w^c - g_{aj}^i w^a v^j \right) \frac{\partial}{\partial v^i} . \quad (2)$$

Making the following identifications:

$$\mu_1(\lambda_i) = t_i^a \tau_a , \quad \mu_2(\tau_a, \tau_b) = f_{ab}^c \tau_c , \quad \mu_2(\tau_a, \lambda_i) = g_{ai}^j \lambda_j ,$$

$$\mu_3(\tau_a, \tau_b, \tau_c) = h_{abc}^i \lambda_i ,$$

for $\{\tau_a\}$ spanning W and $\{\lambda_i\}$ spanning V , one can easily check that requiring $Q^2 = 0$ yields the correct higher Jacobi identities.

2.2. Multisymplectic spaces

Let us now look at a specific example of a realisation of these higher homotopy structures. To do this, we look at a higher analogue of the symplectic structure on even-dimensional manifolds and the Lie algebra product it induces.

Definition 2.3. A *multisymplectic manifold*, or *n-plectic manifold*, is a manifold M endowed with an $(n + 1)$ -form ω that is

- closed, *i.e.* $d\omega = 0$;
- non-degenerate, *i.e.* $\iota_X \omega = 0 \Leftrightarrow X = 0$, where X is a vector field on M .

Just like in symplectic geometry, we can use this structure to define *Hamiltonian vector fields* X_α corresponding to $(n - 1)$ -forms α via

$$d\alpha = -\iota_{X_\alpha} \omega .$$

This immediately suggests how to define higher order products (cf.¹⁰⁻¹²):

Definition 2.4. The *strong homotopy algebra of local observables* (or *shlalo* for short) of (M, ω) , denoted by Π_n , is given by the vector-space complex

$$L : C^\infty(M) \xrightarrow{\pi_1} \Omega^1(M) \xrightarrow{\pi_1} \dots \xrightarrow{\pi_1} \Omega^{n-1}(M) ,$$

together with the brackets for $f \in C^\infty(M)$, $\alpha_i \in \Omega^{n-1}(M)$:

$$\pi_1(f) = df , \quad \pi_k(\alpha_1, \dots, \alpha_k) = (-1)^{\binom{k+1}{2}} \iota_{X_{\alpha_1}} \dots \iota_{X_{\alpha_k}} \omega .$$

$\Pi_n = (M, \omega, \pi_k)$ is clearly a Lie n -algebra. It is not exactly the higher analogue of a Poisson algebra, because there is no obvious way to define an associative product between observables (between two 1-forms, for example) that respects the product structure. Nonetheless, from a physics point of view, it is of course tempting to expect these objects to be the 'classical limit' of the n -algebra structure of some quantum theory. Alternatively, one might expect Π_n to be the starting point to quantizing an $(n + 1)$ -dimensional world-volume. Indeed, one can relate these n -algebras to Nambu-Poisson structures of rank $(n + 1)$ (see¹³). For completeness' sake, let us note that Nambu-Poisson structures themselves are expected to quantize to Lie n -algebras, albeit in a quite complicated way (see¹⁴ and the references therein). One type of structure that was hoped to encode the quantum behaviour of extended objects was the triple bracket introduced by Bagger, Lambert and Gustavsson to describe stacks of M2-branes^{15,16}. These *BLG 3-Lie-algebras* are in fact *strict* Lie 2-algebras (*i.e.* they have $\mu_3 = 0$, so they are differential crossed modules of pairs of actual Lie algebras), so the strong homotopy algebra language may be the correct approach for the necessary generalizations. Interestingly, the ABJM model for M2-branes¹⁷ can also be shown to be a higher gauge theory¹⁸.

We will show an explicit example of a BLG 3-Lie algebra and its corresponding strict 2-algebra in the last section, but for a more general discussion of the correspondence we recommend¹⁹.

2.3. Symplectic NQ -manifolds

We are still missing a fundamental ingredient for the construction of physical actions: that is an invariant metric via which to pair Lie n -algebra valued fields. Such an inner product is best introduced in the NQ -manifold framework.

Definition 2.5. A *symplectic NQ -manifold* is an NQ -manifold (\mathcal{M}, Q) endowed with a closed, non-degenerate, 2-form ϖ , of “ghost” degree $p = n + 1$, invariant under Q :

$$\varpi = \frac{1}{2} \varpi_{AB} dZ^A \wedge dZ^B, \quad \text{s.t.} \quad \mathcal{L}_Q \varpi = 0.$$

Again, we have denoted with Z^A the coordinates on \mathcal{M} and \mathcal{L} indicates the Lie derivative. As usual, n here is the degree of the highest weight coordinate on \mathcal{M} .

Since ϖ is non-degenerate, its inverse can be used to induce a bilinear graded symmetric inner product: $\{-, -\}_\varpi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$. As in symplectic geometry, each function F on \mathcal{M} has a corresponding vector defined by $dF = -\iota_{V_F} \varpi$, and one sets

$$\{F, G\}_\varpi := \iota_{V_F} \iota_{V_G} \varpi.$$

This structure also allows us to find the “Hamiltonian” \mathcal{S} associated to the nilpotent Q operator, since $Q(F) = \{\mathcal{S}, F\}_\varpi$, which squares to zero in the bracket: $\{\mathcal{S}, \mathcal{S}\}_\varpi = 0$.

On the dual side, for the Lie n -algebra L defined by the symplectic NQ -manifold, ϖ translates to a metric on the vector space, $(-, -) : L \times L \rightarrow \mathbb{R}$. In particular, for $l_k \in L$, it will have the following symmetry and invariance under the k -ary products:

$$(l_1, l_2) = (-1)^{|l_1|+|l_2|} (l_2, l_1),$$

$$(\mu_k(l_1, \dots, l_k), l_0) = (-1)^{k+|l_0|(|l_1|+\dots+|l_k|)} (\mu_k(l_0, \dots, l_{k-1}), l_k).$$

This is usually referred to as a *cyclic metric* (for more details and the original reference see^{7,20–22}), while $(L, (-, -))$ is now a *metric Lie n -algebra*.

We now have all the elements in our mathematical tool-box to start constructing gauge-theory actions. We will be dealing with multiplets of gauge

connections, valued in truncated L_∞ -algebras L endowed with a cyclic metric induced by ϖ . The latter, being compatible with the Q -structure on the space dual to L , allows us to easily select only gauge-invariant objects, but also to then apply variational principles to our model.

3. Topological n -algebra models

3.1. Construction

We often refer to the following as a generalization of the AKSZ method for constructing actions²³, but it is in fact inspired by the work of Atiyah²⁴ and later applications by various authors^{25–29}. We start with the following diagram:

$$\begin{array}{ccc} & & T[1]\mathcal{M} \\ & \nearrow f & \downarrow \pi \\ T[1]\Sigma & \xrightarrow{a} & \mathcal{M} \end{array}$$

of an NQ -manifold \mathcal{M} , representing the internal symmetry algebra of our theory, its tangent bundle $T[1]\mathcal{M}$ and the tangent bundle of space-time $T[1]\Sigma$. The number $[k]$ in square brackets indicates a shift in the degree of the coordinates of that space. Each of these spaces comes with a Q -structure: $Q_\Sigma = d_\Sigma$ on the space-time and $Q_\mathcal{M}$ will be the usual dual, or higher Chevalley-Eilenberg, operator on a Lie n -algebra (see definition (2)). On $T[1]\mathcal{M}$ there are two degree-1 differential maps, whose sum gives the full operator: $Q_{T\mathcal{M}} = \bar{d} + \mathcal{L}_{Q_\mathcal{M}}$, where \bar{d} is just a degree-shift operator. The degree-preserving map $a : T[1]\Sigma \rightarrow \mathcal{M}$ will be referred to as a connection. The map f has been introduced because, as opposed to a , it does commute with the Q -structures and is therefore a Q -morphism:

$$f^*(\pi^*h) = a^*(h) \ , \quad \text{and} \quad f^*(\bar{d}h) = (d_\Sigma \circ a^* - a^* \circ Q_\mathcal{M})(h) \ ,$$

where $h \in C^\infty(\mathcal{M})$. Clearly we have $f^* \circ Q_{T\mathcal{M}} = Q_\Sigma \circ f^*$. Explicitly, for the coordinate Z^K on \mathcal{M} , let us call

$$\mathcal{A}^K = \frac{1}{K!} \mathcal{A}_{\mu_1 \dots \mu_K}^K dx^{\mu_1} \wedge \dots \wedge dx^{\mu_K} := a^*(Z^K) \ ,$$

where we have called the weight $[Z^K] \equiv K$, so that Z^K pulls back to a K -form. The pullback along f of $\bar{d}Z^K$, or, equivalently, the failure of a to be a Q -morphism, then gives the *higher (fake) curvature* of \mathcal{A}^K :

$$\mathcal{F}^K = \frac{1}{(K+1)!} \mathcal{F}_{\mu_1 \dots \mu_{K+1}}^K dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{K+1}} := (d_\Sigma \circ a^* - a^* \circ Q_\mathcal{M})(Z^K) \ , \quad (3)$$

which has form-degree one more than the weight of the coordinate Z^K .

Going back to our 2-algebra example, with $\mathcal{M} = W[1] \oplus V[2]$, parametrized by $\{w^a\}$, $\{v^i\}$, graded 1 and 2 respectively, we have

$$a^*(w^a) = A^a = A_\mu^a dx^\mu, \quad a^*(v^i) = B^i = \frac{1}{2} B_{\mu\nu}^i dx^\mu \wedge dx^\nu,$$

so that $\mathcal{A} = (A, B)$ form a 2-connection, with (fake) curvatures

$$\begin{cases} \mathcal{F}^a = d_\Sigma A^a + \frac{1}{2} \mu_2(A, A)^a - \mu_1(B)^a \\ \mathcal{F}^i = d_\Sigma B^i + \mu_2(A, B)^i - \frac{1}{6} \mu_3(A, A, A)^i \end{cases}.$$

We would now like to write down actions with fields valued in a Lie n -algebra, that are invariant under the internal symmetry. This ultimately means that we are looking to pull back Q -invariant polynomials on $T[1]\mathcal{M}$ to our space-time manifold $T[1]\Sigma$. We further want to respect the cohomology from the n -algebra side, that is we want $Q_{T\mathcal{M}}$ -exact terms to pull back to d_Σ -exact objects. The most obvious exact invariant polynomial on $T[1]\mathcal{M}$ is of course its symplectic structure ϖ . On $T[1]\mathcal{M}$ it is given by $\varpi = \varpi_{AB} Q_{T\mathcal{M}} Z^A Q_{T\mathcal{M}} Z^B = \varpi_{AB} (\bar{d} + Q_{\mathcal{M}}) Z^A (\bar{d} + Q_{\mathcal{M}}) Z^B$. Let us consider its “non-covariant” version, $\hat{\varpi} = \hat{\varpi}_{AB} \bar{d} Z^A \bar{d} Z^B$, which is also $Q_{T\mathcal{M}}$ -invariant (since $\iota_Q \varpi = \iota_Q \hat{\varpi} = -d\mathcal{S}$, is exact). Because of the definition of higher curvature from above, it is clear that this object should pull back to the product of \mathcal{F}^a and \mathcal{F}^i . So let us look for the “potential” giving rise to this polynomial: some χ , such that $\hat{\varpi} = (\bar{d} + \mathcal{L}_{Q_{\mathcal{M}}})\chi$. We again want $Q_{\mathcal{M}}$ -invariance, which translates to requiring that if χ restricted to \mathcal{M} (projected via π) is given by some function $\kappa \in C^\infty(\mathcal{M})$, then $Q_{\mathcal{M}}(\kappa) = 0$. It can be easily verified that this potential χ is given by

$$\chi = \varpi_{AB} Z^A \bar{d} Z^B - \mathcal{S},$$

which is referred to as the *Chern-Simons element*, for topological field theories. Under the pullback f^* , this element will give us an action invariant under the gauge n -algebra:

$$S = \int_\Sigma f^* \chi = \int_\Sigma [(\mathcal{A}, \mathcal{F}) + a^*(\mathcal{S})],$$

where we recall that $(-, -)$ is the cyclic metric on the algebra induced by ϖ , and \mathcal{S} is the “Hamiltonian” to $Q_{\mathcal{M}}$. Regrouping all the higher gauge connections into a single field

$$\phi = \sum_A (\pm a^*(Z^A)),$$

where the signs can be freely chosen or reabsorbed into the fields, the equations of motion will be given by

$$d_{\Sigma}\phi + \sum_k \frac{(-1)^{\sigma_k}}{k!} \mu_k(\phi, \dots, \phi) = 0 ,$$

where the signs σ_k will depend on the choices made for ϕ . These can also be written as just $\mathcal{F}^A = 0$, for each degree of the components of \mathcal{M} , *i.e.* they are just *higher flatness conditions*, also referred to as the *higher Maurer-Cartan equations* (for more details on this set-up, see¹³). As it happens, these are also the conditions that category theoretical considerations require so as to have a well-defined concept of higher holonomy on n -dimensional world-volumes, as mentioned in the introduction. They are therefore considered to be of fundamental importance for any consistent higher gauge theory.

3.2. Solutions

So far, everything has been very abstract, as we have been dealing in formal products μ_k and general n -algebras. We have reached the higher flatness conditions, but we do not as yet have any more intuitive picture of what is going on. This is where the n -plectic spaces we discussed in section 2.2 will come in useful. First, however, let us see how to go about solving our higher Chern-Simons models.

From the study of the IKKT model (see^{30,31}), we know that the 0-dimensional reduction of 10-dimensional SYM theory, as a matrix model, looks like a naïve quantization of type IIB string theory, when written in a particular gauge. The embedding coordinates X^μ quantize to the matrix valued fields A^μ , that are to satisfy variational equations of the type $[A_\mu, [A^\mu, A^\nu]] = 0$. Such conditions are clearly solved by the Moyal plane \mathbb{R}_θ^{2k} :

$$[A^\mu, A^\nu] =: [\hat{X}^\mu, \hat{X}^\nu] \sim \theta^{\mu\nu} ,$$

for a constant θ , that is, by quantized embedding coordinates that satisfy the Heisenberg algebra. This kind of solution is to be interpreted as a quantum space-time emerging out of the non-perturbative model, carrying with it the information about how non-commutative the geometry is that the string sees at a high enough energy limit. We follow exactly the same philosophy for our higher gauge theories: we consider our higher CS-theory as if it were the analogue of SYM, as the effective theory for stacks of

branes^b; we reduce our model to 0 dimensions, expecting it to be the high-energy limit, non-perturbative version of some classical theory for extended objects; we thus check if the Π_n version of a higher Poisson algebra of an $(n+1)$ -dimensional object respects the reduced equations of motion of our theory.

From the previous subsection, it is easy to read off the equations of motion for a Lie 2-algebra model, reduced to zero dimensions:

$$\begin{aligned}\mathcal{F}_{ij}^0 &= \frac{1}{2}\mu_2(A_i, A_j) - \mu_1(B_{ij}) \stackrel{!}{=} 0 \\ \mathcal{F}_{ijk}^0 &= \frac{1}{6}\mu_3(A_i, A_j, A_k) + \mu_2(A_i, B_{jk}) \stackrel{!}{=} \epsilon_{ijk} ,\end{aligned}\tag{4}$$

where we allow the 3-form curvature not to vanish, because this condition is not actually needed for well-defined holonomy on a 2-dimensional surface. Just like in non-commutative Yang-Mills theory, this *twist* of the homotopy Maurer-Cartan equations allows for interesting non-commutative solutions, as we will see now.

Consider the shlalo of \mathbb{R}_ω^3 : on 3-dimensional space, a 2-plectic form is obviously given by the volume form

$$\omega = \text{dvol} = \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k ,$$

while the shlalo products are given by

$$\pi(f) = df , \quad \pi_2(\alpha, \beta) = \iota_{X_\alpha} \iota_{X_\beta} \omega , \quad \pi_3(\alpha, \beta, \gamma) = \iota_{X_\alpha} \iota_{X_\beta} \iota_{X_\gamma} \omega .$$

If we choose as a basis of Hamiltonian vector fields $X_{A_i} = \frac{\partial}{\partial x^i}$, the corresponding Hamiltonian 1-forms are given by

$$A_i = \frac{1}{2}\epsilon_{ijk}x^j dx^k .$$

We also need a basis for the functions, which here come from the pullback of degree 2 objects, so they can be given by the 2-forms $B_{ij} = \epsilon_{ijk}x^k$. With these choices, it is easy to verify that $(\mathbb{R}_\omega^3, \omega, \Pi_2)$ solves the equations of motion (4).

This example can be easily generalized to higher n , higher dimensional spaces but also different non-commutative geometries, if we allow the action to contain other gauge-invariant “deformation terms” (some examples for more heuristically constructed actions can be found in⁸). Furthermore, the 0-dimensional actions can be expanded around the solutions and give back

^bOf course we are talking about a topological theory versus SYM, but we are only after a very simple toy-model analysis, to highlight some interesting features of the models, not an actual higher analogue of YM-theory.

what will look like BF-theory on the non-commutative background (again, see⁸ for a detailed example).

4. Relation to Courant algebroids

Earlier we saw how n -plectic manifolds can carry a Lie n -algebra, as an explicit example of how these structures might appear in the physics of extended objects. It may not be too surprising to find that another “generalized” structure, introduced for the study of the novel symmetries seen by 1-dimensional strings, that is T-duality, is also just another example of a higher algebraic structure. Indeed, we will see in what follows how the Courant algebroid, one of the salient features of generalized complex geometry³² and of double field theory³³, can be seen as the Lie 2-algebra carried by a particular Lie 2-algebroid.

4.1. Courant algebroids as NQ -manifolds

It was first shown in³⁴ that the Courant algebroid structure is an example of a Lie 2-algebra. In the following we give a derivation of this fact via the NQ -manifold language (as was done in³⁵), which is powerful because it is easily generalizable to higher dimensional differential forms (rather than just the 1-forms from $TM \oplus T^*M$), but also because it can be used to construct actions via NQ -manifold morphisms, as we will see in section 5.1.

Consider the symplectic NQ -manifold $\mathcal{M} = T^*[2]T[1]\Sigma$, for some degree 0 manifold Σ . We denote the coordinates on \mathcal{M} by $(x^\mu, \xi^\mu, \xi_\mu, p_\mu)$ so that their weights are $(0, 1, 1, 2)$ respectively, making this a Lie 2-algebroid, with non-vanishing *body* Σ of degree 0. The non-degenerate symplectic structure and the nilpotent Q will be given by

$$\begin{aligned}\varpi &= dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\xi_\mu, \\ Q &= \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \xi_\mu} + \frac{1}{2} H_{\mu\nu\rho} \xi^\mu \xi^\nu \frac{\partial}{\partial \xi_\rho} + \frac{1}{3!} \partial_\mu H_{\nu\kappa\lambda} \xi^\nu \xi^\kappa \xi^\lambda \frac{\partial}{\partial p_\mu},\end{aligned}$$

where H is a closed 3-form introduced for generality. One could even introduce more structure, *e.g.* a 3-vector $\tilde{H}^{\mu\nu\rho}$, or various mixed tensors, but these all go beyond the scope of our present analysis.

Consider now the functions over the degree-1 component of \mathcal{M} :

$$e := X^\mu \xi_\mu + \alpha_\mu \xi^\mu \in C^\infty(M_1),$$

and define a metric on this space via the Poisson bracket induced by ϖ :

$$(e_1, e_2) := \frac{1}{2} \{e_1, e_2\}_\varpi = \frac{1}{2} (X^\mu \beta_\mu + Y^\mu \alpha_\mu),$$

for $e_1 = X + \alpha$ and $e_2 = Y + \beta$. At this point, it is worth using the identifications $\xi^\mu \sim dx^\mu$ and $\xi_\mu \sim \partial_\mu$, to make explicit how $e = X + \alpha \in T\Sigma \oplus T^*\Sigma$. The above metric therefore describes the usual pairing from generalized complex geometry:

$$(X + \alpha, Y + \beta) = \frac{1}{2} (\iota_X \beta + \iota_Y \alpha) .$$

It so happens that we can also introduce here an antisymmetric product on M_1 , constructed with the Hamiltonian function Θ corresponding to Q itself^c:

$$\mu_2(e_1, e_2) = \frac{1}{2} (\{\{\Theta, e_1\}, e_2\}_\varpi - \{\{\Theta, e_2\}, e_1\}_\varpi) ,$$

which can be easily verified to translate to

$$\mu_2(X + \alpha, Y + \beta) = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha) + \iota_X \iota_Y H ,$$

that is the antisymmetric version of the twisted Courant bracket for the Courant algebroid $T\Sigma \oplus T^*\Sigma$. We have kept antisymmetry here, at the cost of the Jacobi identity, since here

$$\mu_2(e_1, \mu_2(e_2, e_3)) + \text{cycl.} = \frac{1}{3} ((e_1, \mu_2(e_2, e_3)) + \text{cycl.}) =: d(\mu_3(e_1, e_2, e_3)) .$$

That is, the associativity of the algebra product is violated by an exact term, the argument of which we call μ_3 . In addition, we can call μ_1 the action of Q on $f(x) \in C^\infty(M)$:

$$\mu_1(f(x)) := \{\Theta, f(x)\}_\varpi = df(x) .$$

It can be checked that the products defined in this way do form a Lie 2-algebra structure on the complex $L : C^\infty(\Sigma) \rightarrow C^\infty(M_1)$. One can also easily verify that these products satisfy all the properties defining the exact twisted Courant algebroid $(T\Sigma \oplus T^*\Sigma, \mu_2(-, -), \pi, H)$, where π indicates the algebroid's anchor map, that is the obvious projection to Σ , and H is the twisting 3-form. As we mentioned, we could include more general twists (multi-vectors or mixed tensors), if needed. It is further worth noting that this same discussion can be repeated for higher Lie n -algebroids of the type $T^*[n]T[1]\Sigma$, carrying higher Lie n -algebras, that will contain the Vinogradov algebroid structures on spaces such as $\Lambda^{n-1}T^*\Sigma \oplus T\Sigma$ ^{36,37}. These are of interest when studying M-theory via exceptional generalized geometry, when one wants to include 2- and 5-forms as fundamental objects (see, for instance^{38,39}).

^cAs usual, we can define $QF = \{\Theta, F\}_\varpi$, so that $\{\Theta, \Theta\}_\varpi = 0$.

4.2. Twisted Courant algebroids and n -plectic spaces

We have now discussed two types of structures on potential space-time manifolds M that are somewhat new in theoretical physics, both born out of a need to model certain properties of extended objects: n -plectic spaces and exact Courant algebroids. Both were seen to be examples of specific n -algebras. We will see now that they are in fact related, since the latter contain the structure of the former, as was shown in⁴⁰.

Let us look again at the shlalo Π_2 of a 2-plectic manifold M :

$$\begin{aligned} C^\infty(M) &\xrightarrow{\pi_1} \Omega^1(M) \quad \text{with} \quad \pi_1 = d, \\ \pi_2(\alpha, \beta) &= -\iota_{X_\alpha} \iota_{X_\beta} \omega = \frac{1}{2}(\iota_{X_\alpha} d\beta - \iota_{X_\beta} d\alpha), \\ \pi_2(\alpha, f) &= 0, \quad \pi_3(\alpha, \beta, \gamma) = \iota_{X_\alpha} \iota_{X_\beta} \iota_{X_\gamma} \omega. \end{aligned}$$

As with Lie algebras, we can define structure-preserving n -morphisms for n -algebras (again, see⁴⁰ for the $n = 2$ case, and⁴¹ for the general discussion): in particular, there is a Lie 2-algebra isomorphic to the shlalo described above, whose products are modified to the following:

$$\begin{aligned} \pi_2(\alpha, \beta) &= \frac{1}{2}(\iota_{X_\alpha} d\beta - \iota_{X_\beta} d\alpha) + d(\iota_{X_\alpha} \beta - \iota_{X_\beta} \alpha), \\ \pi_2(\alpha, f) &= \iota_{X_\alpha} df, \quad \pi_3(\alpha, \beta, \gamma) = \iota_{X_\alpha} \iota_{X_\beta} \iota_{X_\gamma} \omega. \end{aligned}$$

Going back to the Lie 2-algebra corresponding to the Courant algebroid: consider those degree-1 functions $e = X_\alpha + \alpha \in C^\infty(M_1)$ whose vector field X_α is precisely the Hamiltonian vector field corresponding to the 1-form α , via the 2-plectic structure $\omega = H$, the twisting 3-form of the Courant bracket. Under this restriction, the Courant 2-algebra yields precisely the above modified Π_2 . Since the 2-plectic structure H needs to be non-degenerate, for the shlalo to make sense, we cannot 'morph away' the twist 3-form, but it now allows for a more geometric interpretation.

It is also worth noting that all of the above discussion can be generalized to higher n , to Π_n on higher dimensional spaces and its relation to the Vinogradov algebroids on $T^*[n]T[1]M$, as is shown in⁴¹. Tying up all these geometric and algebraic structures under the one common theme of categorified algebras may lead to a deeper understanding, and/or easier manipulation, in the context of double field theory and possibly exceptional generalized geometry in M-theory.

5. Example: effective M5-brane dynamics

So far we have introduced the mathematical tools to describe gauge theory based on Lie n -algebra internal symmetries, and we have seen how some

higher structures appear in geometry. We will now combine the two: we construct and examine the equations of motion (*i.e.* the higher Maurer-Cartan equations) of a Lie 2-algebra model, but rather than just on $T[1]\Sigma$, we will have it living on the generalized space-time bundle $T^*[2]T[1]\Sigma$. That is, our world-volume itself is now a graded NQ -manifold, with its own higher Q -structure (as opposed to just $Q_\Sigma = d_\Sigma$). Interestingly, this approach contains the same fields and yields the same dynamics as the proposal in⁴ for the effective action of M5-branes. What follows is a review of the detailed analysis presented in⁴².

5.1. Higher gauge theory on $T\Sigma \oplus T^*\Sigma$

To deduce the higher fake curvatures, we use the procedure we elucidated in section 3.1. We are now looking at the diagram

$$\begin{array}{ccc} & & T[1]L[1] \\ & \nearrow f & \downarrow \pi \\ T^*[2]T[1]\Sigma & \xrightarrow{a} & L[1] \end{array}$$

where we have the usual Lie 2-algebra $L[1] : W[1] \xleftarrow{\mu_1} V[2]$, with its Q_L -structure:

$$Q_L = \left(-\frac{1}{2} f_{bc}^a w^b w^c - t_i^a v^i \right) \frac{\partial}{\partial w^a} + \left(\frac{1}{6} h_{abc}^i w^a w^b w^c - g_{aj}^i w^a v^j \right) \frac{\partial}{\partial v^i} .$$

On the space-time side we use the untwisted Q_C -structure

$$Q_C = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \xi_\mu} ,$$

which, we recall from the previous section, gives the (untwisted) Courant algebroid structure to $T[1]\Sigma \oplus T^*[1]\Sigma$. We now have coordinates of degrees 1 and 2 respectively, (w^a, v^i) , on the $L[1]$ side, and of degrees 0, 1 and 2, $(x^\mu, \xi^\mu, \xi_\mu, p_\mu)$, on the Σ side. That is, when using the grade preserving pullback a^* , the most general 2-connection we obtain will be given by

$$\begin{aligned} A^a &= a^*(w^a) = A_\mu \xi^\mu + A^\mu \xi_\mu , \\ B^i &= a^*(v^i) = \frac{1}{2} B_{MN} \xi^M \xi^N + B^\mu p_\mu , \end{aligned} \tag{5}$$

where the capital indices M, N indicate both up and down μ indices, for compactness. We immediately notice the vector field B^μ at degree 2: its natural appearance in this framework is important to the understanding of the proposal in⁴, where such a field is necessary, for the consistent behaviour

of the theory under dimensional reduction, but it is ultimately added in by hand.

Recall that the higher curvatures were defined by the failure of a to be a Q -morphism, as in eq. (3). Applying this here, one obtains:

$$\begin{aligned}\mathcal{F}^a &= [\partial_M A_N + \tfrac{1}{2}\mu_2(A_M, A_N) + \tfrac{1}{2}\mu_1(B_{MN})] \xi^M \xi^N + (A^\mu + \mu_1(B^\mu)) p_\mu \\ \mathcal{F}^i &= [-\tfrac{1}{6}\mu_3(A_M, A_N, A_K) + \tfrac{1}{2}\mu_2(A_M, B_{NK}) + \tfrac{1}{2}\partial_M B_{NK}] \xi^M \xi^N \xi^K \\ &\quad + (\mu_2(A_\mu, B^\nu) + B_\mu^\nu + \partial_\mu B^\nu) \xi^\mu p_\nu + (\mu_2(A^\mu, B^\nu) + \tfrac{1}{2}B^{\mu\nu}) \xi_\mu p_\nu, \end{aligned} \quad (6)$$

for the part of the curvature \mathcal{F}^a valued in $W[1]$ and \mathcal{F}^i in $V[2]$. Again, the capital indices M, N run over upper and lower μ indices, that is over all of $T[1]\Sigma \oplus T^*[1]\Sigma$.

We saw that the obvious topological higher gauge theory action requires the higher curvatures to vanish, as its equations of motion. We further know that this requirement really underlies the whole motivation for the framework, since it is the only way to guarantee a well-defined holonomy for extended objects. Indeed, it can be shown that the equations of motion of various known supersymmetric theories can be expressed as the vanishing of appropriately identified higher curvatures (for detailed examples see¹³). It is therefore reasonable to take this as the fundamental guiding principle, even in the absence of an explicitly written action. In the following subsection we will see how the above described *generalized 2-gauge theory*, via the zero-curvature principle, reproduces precisely the equations of motion proposed by Lambert and Papageorgakis for the effective dynamics of M5-branes.

5.2. Effective dynamics of M5-branes

Let us start by quickly reviewing the model proposed in⁴. The field content consists of the 6-dimensional $(2,0)$ -multiplet: 5 scalar fields X^I , antichiral fermions Ψ and the self-dual exact 3-form $h = dB \in \Omega^3(\mathbb{R}^{1,5})$, all valued in \mathbb{R}^4 . Upon reduction along a circle to 5 dimensions, the supersymmetry transformation of these fields should reproduce those of 5-dimensional super-Yang-Mills theory, which include a term of the type $[X^I, X^J]$ for $\delta\Psi$. The Ansatz chosen by the authors is to introduce a new vector field $C = C^\mu \partial_\mu$, also valued in \mathbb{R}^4 , to couple to a term quadratic in X^I in the 6-dimensional $\delta\Psi$, which adjusts for the mismatch in chirality of Ψ and the supersymmetry parameter. The model further contains a gauge potential $A_\mu dx^\mu$, valued in $\mathfrak{so}(4)$.

The internal symmetries \mathbb{R}^4 and $\mathfrak{so}(4)$ arise because the authors' Ansatz is based on a 3-Lie algebra structure^d, by which we mean the ternary brackets first introduced by Bagger, Lambert and Gustavsson (BLG) to model stacks of M2-branes (cf. ^{15,16}). As mentioned earlier, these triple structures are in fact a particular example of Lie 2-algebras: they correspond to *strict* 2-algebras, that is those whose Jacobiator μ_3 is identically vanishing. Symmetry considerations, like the closure of the superalgebra, together with the correct behaviour under dimensional reduction, lead to the following strict Lie 2-algebra for this model: the vector space complex $L : W[1] \leftarrow V[2]$ has $W = \mathfrak{so}(4)$ and $V = \mathbb{R}^4$,

$$L : * \leftarrow \mathfrak{so}(4)[1] \leftarrow \mathbb{R}^4[2] .$$

We are dealing with a strict 2-algebra, so the product μ_2 on $\mathfrak{so}(4)$ is just the Lie bracket $[-, -]_{\mathfrak{so}(4)}$ of the algebra, while the action of $y \in \mathfrak{so}(4)$ on elements in $\chi \in \mathbb{R}^4$, that is $\mu_2(y, \chi)$, is the obvious $\mathfrak{so}(4)$ action on vectors. Before we move to the homotopy product μ_1 , we note that there exists a map $D : \mathbb{R}^4 \wedge \mathbb{R}^4 \rightarrow \mathfrak{so}(4)$ defined via

$$(\langle y, D(\chi_1, \chi_2) \rangle)_{\mathfrak{so}(4)} := (y\chi_1, \chi_2) = -(y\chi_2, \chi_1) ,$$

where $(-, -)$ stands for the metric on $\mathfrak{so}(4)$, $(-, -)$ for that on \mathbb{R}^4 and $\chi_i \in \mathbb{R}^4$ while $y \in \mathfrak{so}(4)$. The map D can be used to construct the antisymmetric BLG ternary product: $[\chi_1, \chi_2, \chi_3]_{\text{BLG}} := D(\chi_1, \chi_2)\chi_3$. It is easy to check that this structure indeed satisfies the *fundamental identity* of BLG 3-Lie algebras (see also ¹⁹ for more details on the relation between BLG 3-algebras and strict 2-algebras).

The conditions for the closure of the superalgebra of the M5-brane model lead to the following equations of motion for the gauge gauge fields:

$$0 = h_{\mu\nu\kappa} - \frac{1}{3!}\varepsilon_{\mu\nu\kappa\rho\sigma\tau}h^{\rho\sigma\tau} , \quad (7)$$

$$0 = F_{\mu\nu} - D(C^\lambda, h_{\mu\nu\lambda}) , \quad (8)$$

$$0 = \nabla_\mu C^\nu = D(C^\mu, C^\nu) , \quad (9)$$

$$0 = D(C^\rho, \nabla_r h_{\mu\nu\lambda}) , \quad (10)$$

where the covariant derivative is given by $\nabla = d + A$ and $F = dA + \frac{1}{2}[A, A]_{\mathfrak{so}(4)}$ is the traditional curvature of A . In the first line we just wrote the self-duality condition of h explicitly: as it turns out, h could only be

^dThe nomenclature can be confusing, so we insist on reminding the reader that BLG 3-Lie algebras are not Lie 3-algebras.

written as dB , for a 2-form $B_{\mu\nu}$, if B lived in a traditional abelian Lie algebra. Here, however, though B is valued in \mathbb{R}^4 , it is part of the more intricate 2-algebra structure and is always acted on by $\mathfrak{so}(4)$ -valued operators, thus carrying the non-abelian structure with it. As a consequence, closure of the superalgebra over-constrains the field and h itself can no longer be exact, or interpreted as the curvature of some 2-form (again, see⁴ for details).

Before returning to this point, however, we take a look at (9): the fact that $D(C, C) = 0$, implies that C^μ factorizes into a c^μ vector on $\mathbb{R}^{1,5}$ and a constant $v \in \mathbb{R}^4[2]$. This means that $D(v, -)$ now only spans an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(4)$, which in turn implies that $A \in \mathfrak{so}(3)$. Moreover, the map $D(v, -)$ can now be interpreted as the homotopy map μ_1 , as it takes elements from $\mathbb{R}^4[2]$ into elements in $\mathfrak{so}(3)[1]$ and is nilpotent. The strict Lie 2-algebra of interest therefore reduces to

$$L : * \xleftarrow{D(v, -)} \mathfrak{so}(3)[1] \xleftarrow{D(v, -)} \mathbb{R}^4[2] .$$

If we return our attention to the non-exact h , it turns out that while it is not the traditional curvature of a 2-form field, it can be re-expressed as a higher curvature. Indeed, introducing a 2-form B , such that $c^\mu B_{\mu\nu} = 0$, defined via

$$h_{\mu\nu\kappa} = \frac{1}{|c|^2} \left(B_{[\mu\nu} c_{\kappa]} + \frac{1}{3!} \varepsilon_{\mu\nu\kappa\lambda\rho\sigma} B^{[\lambda\rho} c^{\sigma]} \right) ,$$

we can write its (strict) 2-curvature:

$$H = dB + \mu_2(A, B) = *H ,$$

which can be checked to be self-dual (cf.¹⁹ for details on this part).

We would like the higher flatness conditions from our generalized 2-gauge theory, as described in section 5.1, to reproduce the equations of motion for the gauge fields here. We know the 2-algebra structure we need for the internal symmetry, while for the space-time side we set $\Sigma = \mathbb{R}^{1,5}$, meaning that the NQ -manifold with the appropriate Courant structure is $T^*[2]T[1]\mathbb{R}^{1,5}$. We can also identify our vector field B^μ arising as a pullback of a degree 2 object in L with the field C^μ appearing in the M5-brane model. This means that we can impose that $D(B^\mu, B^\nu) = 0$. What happens if we set the higher curvatures (6) to zero? Let us start by those components that are proportional to p_μ . Since $\mu_1(B^\mu) \sim D(B^\nu, B^\mu)$, we deduce from the p_μ term in \mathcal{F}^a that $A^\mu = 0$. From the last term in \mathcal{F}^i , this implies that also $B^{\mu\nu} = 0$. If we now look at $(\mathcal{F}^i)_{\mu\nu}{}^\rho$, we see that its vanishing requires $B_\nu{}^\rho$ to be covariantly constant, so that it can be gauged away. Now, the only non-zero components left in $(\mathcal{F}^a)_{MN}$ are $\mathcal{F}_{\mu\nu}^a$, whose vanishing gives

(8). Requiring $(\mathcal{F}^i)_\mu{}^\nu = 0$ is just the first part in equation (9), stating that $\nabla_\mu C^\nu = 0$. The last condition that is left is just the vanishing of the component $(\mathcal{F}^i)_{\mu\nu\rho}$, which is precisely the self-duality condition for h . We have therefore recovered all the equations of motion of the gauge fields of the M5-brane model, using just our guiding principle of setting higher curvatures to zero, albeit, here, with a generalized space-time bundle with its own Lie 2-algebra structure. The novel vector field $C^\mu = B^\mu$ finds a natural role in the higher gauge-theory framework.

6. Conclusions

Recapping, we argue that the fundamental symmetry structure for any type of gauge theory of extended objects is given by truncated L_∞ -algebras, or Lie n -algebras. We have shown how to construct topological higher gauge theories, and we have seen how such models are solved by gauge n -connections that satisfy higher flatness conditions. These conditions are also the ones that guarantee that one can well define a concept of holonomy, or of “surface ordering”, over extended world-volumes. Naturally, this is a fundamental requirement if we want to write down reparametrization invariant theories. We therefore consider this same set of conditions to be our guiding principle for describing any kind of model involving extended fundamental objects, not only topological ones. Indeed, the equations of motion of various supersymmetric theories can be rewritten as higher flatness conditions, as can be seen in¹³. In particular, here we have shown the details of how this works for the M5-brane model proposed by Lambert and Papageorgakis⁴.

This model for M5-branes is actually equivalent to one of D4-branes, as for any choice of vacuum expectation value for C^μ , the total symmetry group breaks down to that expected for 5-dimensional super-Yang-Mills theory. We are not arguing that this discussion be complete, but we find it interesting and possibly quite meaningful that even the simplest higher structure can give insight not only into more complex behaviour of gauge fields, but also into the field content itself necessary for a consistent theory. It is of course seductive to think that the generalized geometry introduced to better understand the dualities of an extended object, such as a string, is just another manifestation of the higher gauge structures underlying a more complete quantum theory.

We showed how, along with Courant algebroids, also the BLG triple brackets, as well as the natural structure on n -plectic manifolds, all fit into

the same higher algebra framework. When looking to quantize extended space-times themselves, we therefore propose that the latter Π_n algebras are the correct objects to be considering, where Nambu-Poisson manifolds could be too restrictive. It seems that the best way to approach this issue is via some higher geometric quantization, as is being investigated by^{40,43–45}.

Overall, the unifying power of the higher gauge algebra framework should definitely be a step forward in the quest for a theory of quantum gravity. Indeed, strong homotopy algebras appear in certain reformulations of gravity⁴⁶, and they are expected to underly spin-foam models^{47–49}. Furthermore, they made one of their first appearances in physics in the context of string field theory²² and it may not be surprising that they may encode the full symmetry structure of higher spin theories as well^{50–52}. We would therefore like to leave the reader with the thought that, possibly, unifying all these approaches into the one appropriate language or formalism, might bring the construction of our scientific tower of Babel one step closer.

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